CONSTRAINED DESIGN OF FIR FILTERS WITH SPARSE COEFFICIENTS

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ABSTRACT

We present an algorithm for the constrained design of FIR filters with sparse coefficients. In general, the filter design approach aims to minimize a filter order and maximize the filter performance. Although the FIR filter coefficients designed by the LS method is optimal in the least squares sense, it is not necessarily optimal among the set of filters with the same number of multipliers, that is, less mean squared error can be achieved by a filter that has the same number of multipliers, but has longer impulse response with some zero-valued entries. Our method minimizes the number of nonzero entries in the impulse response together with the least squares error of its frequency response. We incorporate some constraints to the design and realize better performance than conventional constrained least squares design.

***Index Terms*—** FIR filter design, coefficient sparsity, l_0 approximation

1. INTRODUCTION

The design of FIR filters is an important issue in digital signal processing. Many design methods have been proposed by a number of authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Especially the least-squares (LS) method, which minimizes the mean squared error of frequency responses, is widely used due to its simplicity and flexibility [5, 6, 7, 8, 9, 10].

In general, the filter design approach aims to minimize a filter order and maximize the filter performance. Although the FIR filter coefficients designed by the LS method is optimal in the least squares sense, it is not necessarily optimal among the set of filters with the same number of multipliers, that is, less mean squared error can be achieved by a filter that has the same number of multipliers, but has longer impulse response with some zero-valued entries. To minimize the number of multipliers instead of the filter order, some approaches design the filter with zero-valued taps, which is often called sparse filters [11, 12] (Fig.1). In [12], an efficient design method is proposed. However the method needs to design filters iteratively, and high computational effort is required for high order filters. Our previous paper [13] proposes a design method for sparse FIR filters which achieve a sub-optimal approximation in the least squares sense and applies it to minimax design, but we do not consider any constraints in the design.

In the LS design, large error often occurs near a cut-off frequency. Adam et al. address the problem by adding constraints to the filter design algorithm [14]. In the method, the error is reduced by adding the peak error constraints to its frequency response without having large transition bands. Other than that above, the constrained least squares approximation can realize the flexible design, such as flatness constraint at $\omega = 0$ [15, 16] or time domain constraints (e.g. N-th band constraint [17]).

In this paper, we present a numerical approach for designing the constrained sparse filters. Our method can design FIR filters by solving the optimization problem which is formulated by adding a constraint in the frequency response and a sparsity in the coefficients as regularization terms. Our method does not guarantee any optimality in the sense of sparsity, but has better performance than the conventional filter design methods.

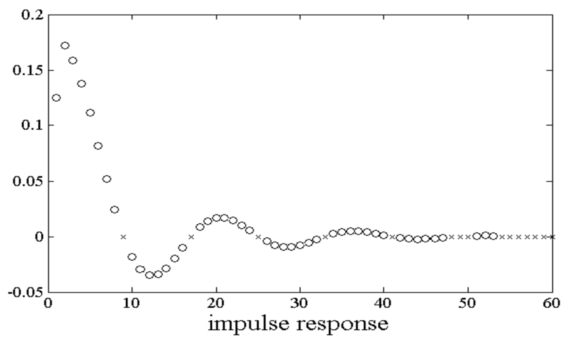


Fig. 1. Impulse response of sparse filters: 'x' indicates zero coefficients.

In Section 2, the conventional weighted least squares and constrained least squares method for the FIR filters are briefly described. In Section 3, our design problem is formulated and a design algorithm that considers the sparsity of coefficients and the constraint for peak errors is proposed. In Section 4, several examples are shown to verify the validity of the proposed algorithm, and some comparisons with the conventional method are shown. In the last section, we describe the advantages of the algorithm and conclude.

2. CONVENTIONAL METHODS

2.1. Weighted Least-Squares Method

The magnitude response $H(\omega)$ of the linear phase FIR filters and more general non-linear phase FIR filters can be written as a linear combination of trigonometric basis functions

$$H(\omega) = \sum_{n=0}^{N-1} a_n \phi_n(\omega), \tag{1}$$

where, for example, $\phi_n(\omega) = \cos(n\omega)$ for even-order symmetric linear phase filters, and $\phi_n(\omega) = e^{jn\omega}$ for more general formula.

Our design method can handle all types of the FIR filters expressed by (1), however due to the limited space, we show only the case of the even-order symmetric linear phase FIR filters.

We here repeat $H(\omega)$ for the even-order symmetric filters:

$$H(\omega) = \sum_{n=0}^{N-1} a_n \cos(n\omega), \tag{2}$$

where $N = (N_0 - 1)/2 + 1$ and N_0 is its filter length.

Here we briefly review the conventional weighted LS (WLS) method. Generally, the mean squared error of (2) over the interval $[0, \pi]$ is defined as

$$\Phi_1 = \frac{1}{\pi} \int_0^{\pi} W(\omega) |H(\omega) - D(\omega)|^2 d\omega, \tag{3}$$

where $W(\omega)$ is a weight function which is not identically zero and has positive values, and $D(\omega)$ is a desired frequency response. The optimal filter coefficients $\{a_n\}_{n=0}^{N-1}$ in the LS sense can be uniquely determined by solving the normal equation, $\mathbf{Qa} = \mathbf{p}$, where $\mathbf{a}$ is a filter coefficient vector $\mathbf{a} = [a_0 \ \ a_1 \ \ \cdots a_{N-1}]^T$, and

$$\begin{aligned} p_m &= \tfrac{1}{\pi} \textstyle\int_0^{\pi} W(\omega) D(\omega) \cos(m\omega) d\omega, \\ Q_{m,n} &= \tfrac{1}{\pi} \textstyle\int_0^{\pi} W(\omega) \cos(m\omega) \cos(n\omega) d\omega, \end{aligned} \tag{4}$$

where p_m and $Q_{m,n}$ are m-th and (m, n)-th elements of $\mathbf{p}$ and $\mathbf{Q}$, respectively.

When $D(\omega)$ and $W(\omega)$ are simple functions, Eqs. (4) can be easily calculated in a closed form. However, commonly, the integrals of (4) are difficult to derive if both $D(\omega)$ and $W(\omega)$ are arbitrary. Therefore, in practice, it is efficient to define the following cost function which is expressed as the finite sum of the errors on the discretized frequency points.

$$\begin{aligned} \Phi_2 &= \frac{1}{L} \sum_{l=0}^{L-1} W(\omega_l) |H(\omega_l) - D(\omega_l)|^2 \\ &= \|\mathbf{W}(\mathbf{Ra} - \mathbf{d})\|_2^2, \end{aligned} \tag{5}$$

where the (l, k)-th elements of $\mathbf{R}$ is

$$\mathbf{R}_{l,k} = \cos(k\omega_l) \qquad \text{and}$$

$$\mathbf{d} = [D(\omega_0) \ \ D(\omega_2) \cdots \ \ D(\omega_{L-1})]^T$$
$$\mathbf{W} = diag\{W(\omega_0) \ \ W(\omega_2) \cdots \ \ W(\omega_{L-1})\}.$$

In this case, p_m and $Q_{m,n}$ of (4) in the normal equation are rewritten as follows.

$$\begin{aligned} p_m &= \tfrac{1}{L} \textstyle\sum_{l=0}^{L-1} W(\omega_l) D(\omega_l) \cos(m\omega_l) \\ Q_{m,n} &= \tfrac{1}{L} \textstyle\sum_{l=0}^{L-1} W(\omega_l) \cos(m\omega_l) \cos(n\omega_l) \end{aligned}$$

This is the most general WLS method for linear phase FIR filters. When $W(\omega)$ is set to unity over $[0, \pi]$, best l_2 filters with this error weighting process (i.e. LS) has large peak errors near the band edges (see Fig.2 (a)). Therefore, $W(\omega)$ is usually set to 0 in the transition bands.

2.2. Orthogonal Matching Persuit

The classical sparse approximation problem is denoted by

$$\arg\min_{\mathbf{s}} \|\mathbf{s}\|_0 \text{ subject to } \|\mathbf{y} - \mathbf{\Phi s}\|_2 < \epsilon,$$

where $\mathbf{s} \in \mathbb{R}^M$, and $\mathbf{y} \in \mathbb{R}^N$ and $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$, $(M > N)$ is an overcomplete dictionary. Its sub-optimal solution can be found by greedy algorithms such as the orthogonal matching pursuit (OMP). The OMP [18, 19] successively selects a column vector of the dictionary $\mathbf{\Phi}$ at each iteration which minimizes the residual approximation error. Letting the set of indices of selected vectors be Γ^n, the algorithm can be summarized as:

1. Set $\mathbf{r}^1 = y$, $\Gamma^1 = \emptyset$ and iterate the following steps.
2. $g_i =< \boldsymbol{\phi}_i, \mathbf{r}^n >$ for $i \in \overline{\Gamma^n}$
3. $i^* = \text{argmax}_i |g_i|$
4. $\mathbf{x}_n = (\mathbf{\Phi}_n^T \mathbf{\Phi}_n)^{-1} (\mathbf{\Phi}_n^T \mathbf{y})$
5. $\mathbf{r}^{n+1} = \mathbf{y} - \mathbf{\Phi}_n^T \mathbf{x}_n$
6. $\Gamma^{n+1} = \Gamma^n \cup i^*$

where $\boldsymbol{\phi}_i$ is the i-th column vector of $\mathbf{\Phi}$, and $\mathbf{\Phi}_n$ is a selected dictionary composed of $\boldsymbol{\phi}_i$ $(i \in \Gamma_n)$. The orthogonalization part in Step 4 can be efficiently calculated by the QR decomposition and Gram-Schmidt orthogonalization for fast implementation. Both the OMP and our proposed algorithms are sub-optimal methods. A major advantage of our method is the ease with which certain constraints can be accounted for in the design. In our method, sparse coefficients can be easily found by transforming a constrained problem to an unconstrained one.

3. DESIGN ALGORITHM

The conventional WLS method is optimal in the LS sense among filters with the same filter length. If one allows a longer filter length with some zero coefficients, it can achieve better approximation than the non-sparse filters with the same number of multipliers. Our goal is to design such sparse filters.

3.1. l_2-l_0 based Approximation

Here we define $\|\mathbf{a}\|_0$ as the number of non-zero elements in $\mathbf{a}$ (It is often called zero-norm or l_0-norm, despite not satisfying the properties of the norm). Our aim is to approximate the desired frequency response by the linear phase FIR filter with minimum number of coefficients. To fulfill it, we define the filter design problem as a process to minimize the cost function:

$$\min_{\mathbf{a}} \frac{1}{2} \|\mathbf{W}(\mathbf{Ra} - \mathbf{d})\|_2^2 + \beta \|\mathbf{a}\|_0, \ \text{ s.t. } \ \mathbf{La} - \mathbf{k} \in S \tag{6}$$

where $\mathbf{d}$ is a desired frequency response and the parameter β is introduced to control the balance between the error of the filter and the number of the coefficients, and $\mathbf{La} - \mathbf{k} \in S$ is a linear constraint added to $\mathbf{a}$, and S is a closed convex set.

We convert (6) to the following unconstrained problem by introducing the indicator function ι_0.

$$\min_{\mathbf{a}} \frac{1}{2} \|\mathbf{W}(\mathbf{Ra} - \mathbf{d})\|_2^2 + \beta \|\mathbf{a}\|_0 + \iota_0(\mathbf{La} - \mathbf{k}), \tag{7}$$

where $\iota_0(\mathbf{x})$ penalizes it if each element x_i of $\mathbf{x}$ is not in the convex set S, that is,

$$\iota_0(\boldsymbol{x}) = \begin{cases} 0, & \text{if } \boldsymbol{x} \in S \\ +\infty, & \text{if } \boldsymbol{x} \notin S \end{cases}. \tag{8}$$

Since the second term of the cost function (7) is a discrete metric, the problem is inherently difficult to solve by conventional methods such as gradient decent based optimization.

We adopt a method based on the variable splitting and quadratic penalty. Introducing auxiliary parameters $\mathbf{s}_1$ and $\mathbf{s}_2$ that correspond the coefficients $\mathbf{a}$ and $\mathbf{La} - \mathbf{k}$, respectively, we convert the original problem (7) to an equivalent form as

$$\begin{aligned} \min_{\mathbf{a}, \mathbf{s}_1, \mathbf{s}_2} \quad & \frac{1}{2} \|\mathbf{W}(\mathbf{Ra} - \mathbf{d})\|_2^2 + \beta \|\mathbf{s}_1\|_0 + \iota_0(\mathbf{s}_2), \\ & \text{s.t. } \ \|\mathbf{a} - \mathbf{s}_1\|_2^2 = 0, \text{ and} \\ & \|\mathbf{La} - \mathbf{k} - \mathbf{s}_2\|_2^2 = 0 \end{aligned} \tag{9}$$

Instead of solving (7), we add the penalty terms $\|\mathbf{a}-\mathbf{s}_1\|_2^2/2$ and $\|\mathbf{La}-\mathbf{k}-\mathbf{s}_2\|_2^2/2$ to the cost. In the end, our design problem is stated as

$$\begin{aligned}\min_{\mathbf{a},\mathbf{s}_1,\mathbf{s}_2} \ & \frac{1}{2}\|\mathbf{W}(\mathbf{Ra}-\mathbf{d})\|_2^2+\beta\|\mathbf{s}_1\|_0+\iota_0(\mathbf{s}_2)+\\ & \frac{\gamma_1}{2}\|\mathbf{a}-\mathbf{s}_1\|_2^2+\frac{\gamma_2}{2}\|\mathbf{La}-\mathbf{k}-\mathbf{s}_2\|_2^2,\end{aligned} \tag{10}$$

where $\gamma_{1,2}$ are weighting parameters that balance the two terms.

3.2. Design Procedure

The strategy of the variable splitting approach is that, we start with initial values for the auxiliary parameters, and then repeatedly solve sub-problems. We describe design algorithms to fulfill the minimization problem (10) hereafter.

We start with initial values for $\mathbf{s}_1^0$ and $\mathbf{s}_2^0$, and then repeatedly solve (10) w.r.t. $\mathbf{a}$, $\mathbf{s}_1$ and $\mathbf{s}_2$ as follows.

$$\begin{aligned}\mathbf{a}^k=\arg\min_{\mathbf{a}} \quad & \frac{1}{2}\|\mathbf{W}(\mathbf{Ra}-\mathbf{d})\|_2^2+\frac{\gamma_1}{2}\|\mathbf{a}-\mathbf{s}_1^{k-1}\|_2^2+\\ & \frac{\gamma_2}{2}\|\mathbf{La}-\mathbf{k}-\mathbf{s}_2^{k-1}\|_2^2,\end{aligned} \tag{11}$$

$$\mathbf{s}_1^k=\arg\min_{\mathbf{s}_1} \ \beta\|\mathbf{s}_1\|_0+\frac{\gamma_1}{2}\|\mathbf{a}^k-\mathbf{s}_1\|_2^2 \tag{12}$$

$$\mathbf{s}_2^k=\arg\min_{\mathbf{s}_2} \ \iota_0(\mathbf{s}_2)+\frac{\gamma_2}{2}\|\mathbf{La}^k-\mathbf{k}-\mathbf{s}_2\|_2^2 \tag{13}$$

The first sub-problem (11) has a simple quadratic form with respect to $\mathbf{a}$. The solution is determined by differentiating (11) w.r.t. $\mathbf{a}$ and setting it to 0, which results in (superscript k is omitted hereafter)

$$\mathbf{a}^*=\mathbf{A}^{-1}\mathbf{b}, \tag{14}$$

$$\text{where} \quad \begin{aligned}\mathbf{A}&=\mathbf{R}^T\mathbf{W}^2\mathbf{R}+\gamma_1\mathbf{I}+\gamma_2\mathbf{L}^T\mathbf{L}\\ \mathbf{b}&=\mathbf{R}^T\mathbf{W}^2\mathbf{d}+\gamma_1\mathbf{s}_1+\gamma_2\mathbf{L}^T(\mathbf{k}-\mathbf{s}_2)\end{aligned}$$

The optimal solution of the second sub-problem (12) is found for each coefficient individually, that is, the problem is equivalent to the minimization of the function $E(s_n)$ for $n=0,1,\cdots,N-1$ as follows

$$\begin{aligned}\min_{s_{1,n}} \quad & E(s_n)=\beta C(s_{1,n})+\frac{\gamma_1}{2}(a_n-s_{1,n})^2,\\ & (n=0,1,\cdots,N-1)\end{aligned} \tag{15}$$

where $s_{1,n}$ is the n-th element in $\mathbf{s}_1$, and the function C that counts the number of non-zero elements is defined as

$$C(s_{1,n})=\begin{cases}0 & \text{if } s_{1,n}=0\\ 1 & \text{if } s_{1,n}\neq 0.\end{cases}$$

The solution is given by the well-known hard shrinkage:

$$s_{1,n}^*=\begin{cases}0, & a_n^2<2\beta/\gamma_1\\ a_n, & \text{otherwise}\end{cases} \tag{16}$$

The solution of (13) is obtained by the projection onto S, which is explained for a specific case in the next section.

We update the three parameters by solving (11)-(13), and $\gamma_{1,2}$ is increased during iterations to ensure the similarity between the auxiliary variables and their corresponding parameters. When the weights $\gamma_{1,2}$ increase, $\mathbf{s}_1$ and $\mathbf{s}_2$ get close to $\mathbf{a}$ and $\mathbf{La}-\mathbf{k}$, respectively, and then the cost function (10) approaches (7).

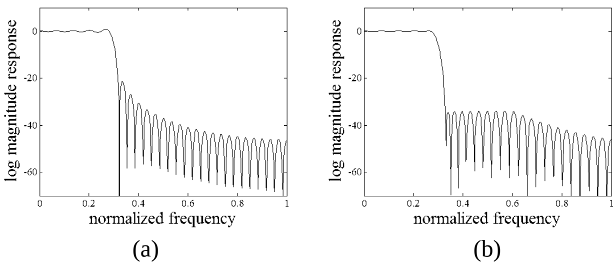


Fig. 2. Example of l_2 filter ($N_0=61$, the cut-off frequency is 0.3π): (a)Best unconstrained l_2 filter, (b)Best constrained l_2 filter with $\delta_p=\delta_s=0.02$

3.3. Constrained Filter Design

3.3.1. Peak Error Constraint

In the filter design, the least squares approximation with the peak error constraint is useful for some applications. Adam et al. [14] proposed a constrained least squares method. In this method, the peak error of the filter can be significantly reduced with only a slight increase in the l_2 error, and later Selesnick et al. [20] proposes a peak constrained design method with adaptive transition bands.

The l_2 problem with the peak error constraint for a lowpass filter with a passband/stopband edges ω_p, ω_s is defined by

$$\begin{aligned}&\min_{\mathbf{a}} \frac{1}{L}\sum_{l=0}^{L-1}W(\omega_l)|H(\omega_l)-D(\omega_l)|^2,\\ &\text{subject to } L(\omega_l)\leq H(\omega_l)\leq U(\omega_l), \omega_l\in[0,\omega_p]\cup[\omega_s,\pi].\end{aligned} \tag{17}$$

Moreover, $L(\omega)$ and $U(\omega)$ are the specified lower and upper bound functions respectively. These functions are given by

$$L(\omega)=\begin{cases}1-\delta_p, & \text{if } \omega\in[0,\omega_p]\\ -\delta_s, & \text{if } \omega\in[\omega_s,\pi]\end{cases} \tag{18}$$

and

$$U(\omega)=\begin{cases}1+\delta_p, & \text{if } \omega\in[0,\omega_p]\\ \delta_s, & \text{if } \omega\in[\omega_s,\pi]\end{cases}, \tag{19}$$

where δ_p and δ_s are the prescribed error bound in the passband and stopband. The peak constrained filters (Fig.2 (b)) are designed by solving (17). (For detail, see [14])

In our case, the sparse filter design with peak error constraints are realized by setting some variables in (6) as

$$\begin{gathered}\mathbf{L}=\mathbf{R}, \ \ \mathbf{k}=\mathbf{0},\\ S=\{\boldsymbol{x}|L(\omega_l)\leq x_l\leq U(\omega_l)\}, \quad l=0,1,\cdots,L-1\end{gathered} \tag{20}$$

where $L(\omega_l)$ and $U(\omega_l)$ are defined by (18) and (19) respectively. In the end, the cost function to minimize is given by

$$\min_{\mathbf{a}} \frac{1}{2}\|\mathbf{W}(\mathbf{Ra}-\mathbf{d})\|_2^2+\beta\|\mathbf{a}\|_0+\iota_0(\mathbf{Ra}), \tag{21}$$

The first two sub-problems (11) and (12) are solved by (14) and (16), respectively. Similarly, the optimal solution of the third sub-problem (13) is found for each frequency point individually, that is,

$$\iota_0\left((\boldsymbol{Ra})_l\right)+\frac{\gamma_2}{2}|(\boldsymbol{Ra})_l-s_{2,l}|^2$$

where $(\boldsymbol{Ra})_l$ is l-th element of the vector $\boldsymbol{Ra}$. One can easily find that the solution for the above problem is given by the projection onto the convex set S in (20) as

$$s^*_{2,n} = \begin{cases} U(\omega_l), & \text{if } (\boldsymbol{Ra})_l > U(\omega_l) \\ L(\omega_l), & \text{if } (\boldsymbol{Ra})_l < L(\omega_l) \\ (\boldsymbol{Ra})_l, & \text{otherwise.} \end{cases} \quad (22)$$

3.3.2. Other constraints

We can easily take into account other linear constraints in our framework. For example, let us consider filters with flatness constraints at a frequency point that have been widely studied [21]. The M-th order flatness constraint is generally given by

$$\frac{d^{(k)}}{d\omega^{(k)}} H(e^{j\omega}) = 0, \quad 1 \leq k \leq M$$

The constraints are easily incorporated into the problem (6). The N-th band filters are widely used to reduce intersymbol interference in data communication [17]. The N-th band filter coefficients have periodic zero values every N-th sample, except for the middle coefficient. This time domain constraint is also expressed by the form of (6).

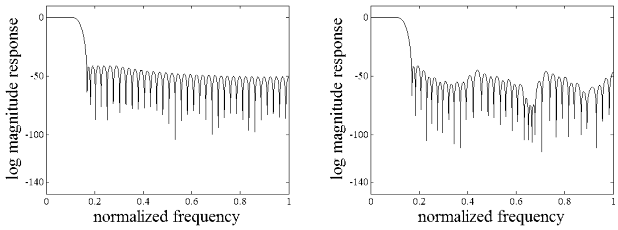
3.4. Balance control

In the algorithm, the desired number of non-zero coefficients (denoted by N_d) is specified by a user. The number of non-zero coefficients is determined by the parameter β in the minimization problem (6). However it is so hard to patently formulate the relationship with β and N_d. Thus we employ a heuristic approach, in which β adaptively changes in the iterations compared with the number of non-zero coefficients. If the actual number of non-zero coefficients in an iteration is larger than N_d, β is increased to $r_u \cdot \beta (r_u > 1)$, otherwise β is decreased to $r_l \cdot \beta (r_l < 1)$, where r_u, and r_l are newly introduced scaling parameters. When the number of the coefficients reaches N_d, then β is fixed.

4. EXPERIMENTAL RESULTS

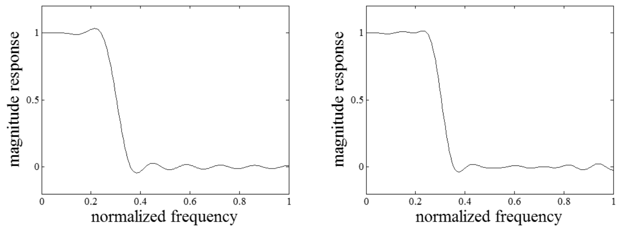
In this section, numerical experiments are shown to verify the advantage of the proposed algorithm. All experiments were designed in MATLAB. All frequencies are normalized by π and frequency points are equally spaced.

A low-pass filter with the cut-off frequency 0.13π is designed. The filter length N_0 is 91 ($N = 45$) and the number of non-zero coefficients is $N_d = 81$. The passband and the stopband edges $\omega_p = 0.112$ and $\omega_s = 0.168$. The result is compared to the 80th order non-sparse filter, which has the same number of multipliers as our filter, designed by the conventional constrained least squares (CLS) approach [14]. In both methods, the constraint parameters are set to $\delta_p = \delta_s = 0.02$. Fig.3 illustrates the log-magnitude response of filters which were designed by (a) the CLS and (b) the proposed method respectively. Note that the error of amplitude with the desired response is within the specified range in both of the methods. We can achieve less mean squared error ($2.31 \cdot 10^{-2}$), compared to the CLS ($3.70 \cdot 10^{-2}$), while the both methods have the same maximum error. We tested hundreds of design examples, and all examples converge to filter with smaller errors than the conventional CLS method. Some of them are listed in Table 1. Table 1 shows that the proposed method outperforms CLS method in the squared error. The result of our experiments clearly shows that the proposed sparse filter approximation method can achieve flexibly design under



(a) (b)

Fig. 3. The log-magnitude response of the obtained filter: (left) CLS, (right) proposed



(a) (b)

Fig. 4. The magnitude response of the obtained filter: (left) Conventional method, (right) Our method

the specified constraint with less l_2 error. In our algorithm the design of 200-tap filter needs ten seconds to converge with Intel Core i5 2.30GHz CPU.

One can easily apply our method to a filter with other constraints. As an example, we show the results of the filter with the 2nd order flatness constraint at $\omega = 0$ in Fig.4 (the linear scale magnitude response is illustrated to show the flatness more clearly), where $N = 41$, $N_d = 31$, and the passband and stopband edges are $\omega_p = 0.26$ and $\omega_s = 0.34$, respectively. Fig.4(a) is the 30th order filter designed by [14] which results in the mean squared error $3.97 \cdot 10^{-4}$, while the error of our result in Fig.4(b) is $1.91 \cdot 10^{-4}$.

5. CONCLUSION

We proposed the flexibility and excellent sparse filter. In the proposed filter, the peak error is specified by the user, and then the better approximation is achieved by the sparsity in the coefficients of the filter. Compared with the conventional constrained filter, the l_2 error of the filter frequency response can be greatly reduced.

Table 1. Squared error (ω_p: passband edge, ω_s: stopband edge, N: filter length, N_d: # of non-zero coeffs., all the two filters listed have same number of non-zero coefficients)

(N_0, $N_0 - N_d$, ω_p, ω_s, δ_p and δ_s)	CLS	Proposed method
(61, 12, 0.264, 0.336, 0.02)	5.08e-3	4.79e-3
(91, 20, 0.1693, 0.2307, 0.009)	3.77e-3	3.51e-3
(121, 24, 0.1760, 0.2240, 0.007)	2.82e-3	2.65e-3
(181, 30, 0.2853, 0.3147, 0.007)	1.78e-3	1.70e-3
(271, 60, 0.2898, 0.3102 0.008)	1.26e-3	1.18e-3
(361, 60, 0.2925, 0.3077 0.007)	8.85e-4	8.64e-4
(481, 84, 0.2943, 0.3057, 0.006)	6.86e-4	6.60e-4
(541, 100, 0.2947, 0.3053 0.005)	6.24e-4	5.75e-4

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